of Eigenvectors," *SIAM Journal on Numerical Analysis,* Vol. 25, June 1988, pp. 679-691.
[8]Sun, J.G., "Eigenvalues and Eigenvectors of a Matrix Dependent on Several Parameters," *Journal of Computational Mathematics,* Vol. 3, Oct. 1985, pp. 351-364.
[9]Sun, J.G. "A Note on Local Behaviour of Multiple Eigenvalues," *SIAM Journal on Matrix Analysis and Applications,* Vol. 10, Oct. 1989, pp. 533-541.
[10]Murthy, D.V. and Haftka, R.T., "Derivatives of Eigenvalues and Eigenvectors of a General Complex Matrix," *International Journal for Numerical Methods in Engineering,* Vol. 26, Feb. 1988, pp. 293-311.
[11]Tan, R.C.E., "Some Acceleration Methods for Iterative Computation of Derivatives of Eigenvalues and Eigenvectors," *International Journal for Numerical Methods in Engineering,* Vol. 28, July 1989, pp. 1505-1519.
[12]Lancaster, P. and Tismenetsky, M., *The Theory of Matrices,* 2nd ed., Academic, New York, 1985, pp. 383-405.

Reply by the Author to A. L. Andrew and K.-W. E. Chu

M. S. Jankovic*
Universe Enterprises Ltd.,
Calgary, Alberta, Canada

IN my original published paper[1] there were several obvious typographical mistakes that the authors of the Technical Comment have correctly identified. These are as follows:

A_Π in Eq. (4) should be replaced by A_π, which is obvious from Eq. (3) and (5); $(P\ Ax)_\pi$ in Theorem VI should be replaced by $(P\,Ax)_\pi$ in Theorem VI should be replaced by $(P\ A_\pi x)$ as can be seen from Eq. (15). The definition of the matrix P is missing in the paper and is given further below.

It is true that the Theorems I to IV are not valid in general for $\alpha = 0$ and their general validity was nowhere stated. They were derived under the assumption that $x_\Pi = -\beta x/2$ [Eq. (13)] which is not true in general. But, regardless of the value of β, Theorems I, II, III and IV are valid if the above assumption is valid; that is if the eigenvector derivative x_Π is proportional to the eigenvector itself in which case $Ax_\Pi = 0$ ($A_\Pi x = 0$). Therefore, for the example given by the authors, Theorems I to IV do apply for the second eigenvalue $\lambda = \pi + 1$ only. The authors failed to mention this. Thus, from Theorem II

$$\begin{aligned}\lambda_\Pi &= x^T A_\pi x \\ &= (x_1 x_2)\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} \\ &= x_2^2\left(\frac{x_1}{x_2}+1\right) \\ &= \frac{1}{(\lambda-\pi)^2+1}(\lambda-\pi+1)\end{aligned}$$

and finally $\lambda_\Pi = 1$. Also from Theorem I, $x_\Pi = 0$.

However, Theorems V and VI for $\alpha \neq 0$ not only appear to be true when $x\dagger Kx_\Pi$ is real, as stated by Andrew and Chu, but are indeed true when $x\dagger Kx_\Pi$ is complex also. Since Andrew and Chu did not include a counter example to disprove Theorems V and VI, I am including a nonlinear example 1), which supports their validity, for a complex matrix A with complex eigenvalues and eigenvectors and complex $x\dagger Kx_\Pi$.

Received Nov. 30, 1980; revision received Jan. 8, 1990.

*Consulting Engineer.

First let me restate Theorems V and VI for completeness.

Theorem V: For a nonlinear eigenvalue problem previously defined, and for $\alpha \neq 0$, the nth derivative of the eigenvalue $\lambda^{(n)}$ with respect to the parameter π is

$$\lambda^{(n)} = 2\left[x\dagger K\left(\frac{PA_\pi}{\alpha}\right)x\right]^{(n-1)}$$

Theorem VI: For a nonlinear eigenvalue problem previously defined, and for $\alpha \neq 0$, the nth derivative of the eigenvector $x^{(n)}$ with respect to the parameter π is

$$x^{(n)} = -(P\,A_\pi x)^{(n-1)} - \frac{(\beta x)^{(n-1)}}{2}$$

where

$$P = (A - \tfrac{2}{\alpha}A_\lambda xx\dagger K)^{-1}$$

is assumed to exist which is for $A_\lambda \neq 0$.

Example 1:

Let

$$A = \begin{bmatrix}c\lambda^2 - \pi\lambda & b\\ a & 1\end{bmatrix}$$

where

λ = complex eigenvalue
π = real parameter
a,b,c = complex constants,

and

$$K = \begin{bmatrix}k & \lambda\\ \lambda^* & k\end{bmatrix}$$

= arbitrary Hermitian matrix

where

k = real constants,
λ^* = complex conjugate of λ.

The problem is to find the derivatives of eigenvalue λ_Π and eigenvector x_Π for a nonlinear eigenvalue problem

$$Ax = 0$$
$$x\dagger Kx = 1$$

From

$$det\ A = c\lambda^2 - \pi\lambda - ab = 0$$

we find by implicit differentiation

$$\lambda_\Pi = \frac{\lambda}{2c\lambda - \pi}$$

and for the eigenvector x

$$x = \begin{bmatrix}x_1\\ x_2\end{bmatrix} = x_1\begin{bmatrix}1\\ -a\end{bmatrix},\quad x_1 = \frac{1}{[k + a^*(ka - \lambda^*) - \lambda a]^{1/2}}$$

the derivatives, calculated by direct differentiation, are

$$x_{1\Pi} = \tfrac{1}{2}x_1^3(a\lambda_\Pi + a^*\lambda_\Pi^*);\ x_{2\Pi} = -\tfrac{1}{2}x_1^3(a^2\lambda_\Pi + |a|^2\lambda_\Pi^*).$$

Now, we use Theorems V and VI to calculate these derivatives. First we calculate

$$A_\lambda = \begin{bmatrix} 2c\lambda - \pi & 0 \\ 0 & 0 \end{bmatrix}; A_\pi = \begin{bmatrix} -\lambda & 0 \\ 0 & 0 \end{bmatrix}$$

$$P = \frac{1}{\Delta} \begin{bmatrix} a_{22} & -a_{12} + \frac{2}{\alpha}(a_{11\lambda}x_1x_1{}^*\lambda + ka_{11\lambda}x_1x_2^*) \\ -a_{21} & a_{11} - \frac{2}{\alpha}(a_{11\lambda}x_1x_1{}^* + \lambda^* a_{11\lambda}x_1x_2^*) \end{bmatrix}$$

where

$$\Delta = -\frac{2}{\alpha}(2c\lambda - \pi)$$

$$\alpha = x\dagger K_\lambda x$$

$$= x_1^* x_2$$

$$\beta = x\dagger K_\pi x$$

$$= x_1 x_2^* \lambda_\Pi^*$$

and $$a_{11\lambda} = 2c\lambda - \pi$$

Then, from Theorem V

$$\lambda_\Pi = \frac{2}{\alpha}(x_1^* \; x_2^*) \begin{bmatrix} k & \lambda \\ \lambda^* & k \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} -\lambda & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= -\frac{2}{\alpha\Delta}\lambda\,|\,x_1\,|^2[k - \lambda a + a^*(ka - \lambda^*)]$$

$$= -\frac{2\lambda}{\alpha\Delta}$$

or

$$\lambda_\Pi = \frac{\lambda}{2c\lambda - \pi}$$

which is the same as obtained by implicit differentiation.

Now, using Theorem VI

$$x_\Pi = -\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} -\lambda & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \frac{\beta}{2}x$$

$$= \lambda x_1 \begin{pmatrix} p_{11} \\ p_{22} \end{pmatrix} + \frac{1}{2}x_1^3 a^* \lambda_\Pi^* \begin{pmatrix} 1 \\ -a \end{pmatrix}$$

$$= \frac{\lambda x_1}{\Delta} \begin{pmatrix} 1 \\ -a \end{pmatrix} + \frac{1}{2}x_1^3 a^* \lambda_\Pi^* \begin{pmatrix} 1 \\ -a \end{pmatrix}$$

and finally

$$x_{1\Pi} = \frac{1}{2}x_1^3(a\lambda_\Pi + a^*\lambda_\Pi^*)$$

$$x_{2\Pi} = -\frac{1}{2}x_1^3(a^2\lambda_\Pi + |a|^2\lambda_\Pi^*).$$

The above derivatives are the same as those obtained by direct differentiation. Furthermore, Eq. (7) in the paper[1]

$$\alpha\lambda_\Pi + 2x\dagger Kx_\Pi = -\beta$$

is satisfied and $x\dagger Kx_\Pi$ is complex.

Reference

[1]Jankovic, M.S., "Analytical Solutions for the *n*th Derivatives of Eigenvalues and Eigenvectors for a Nonlinear Eigenvalue Problem," *AIAA Journal,* Vol. 26, Feb. 1988, pp. 204–205.